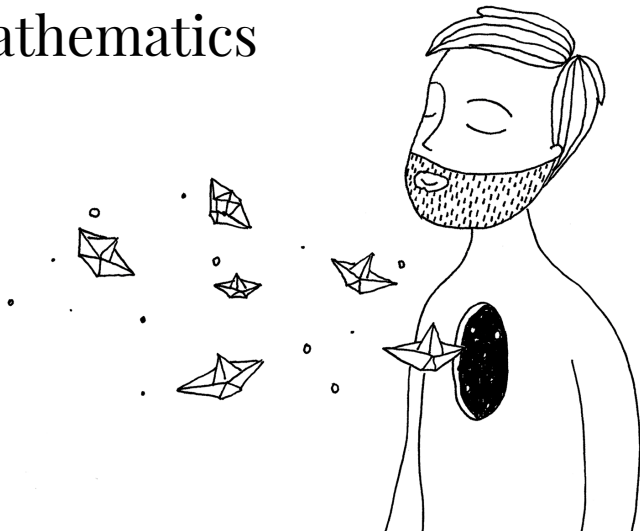


4509 – Bridging Mathematics

Vectors

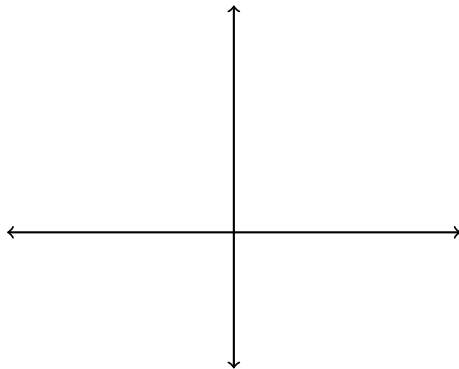
PAULO FAGANDINI

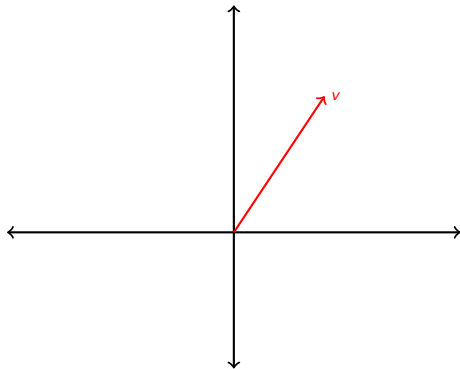


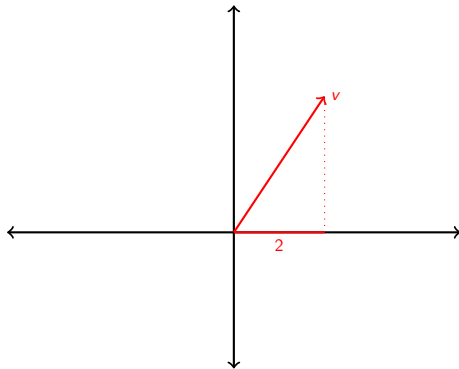
Notation

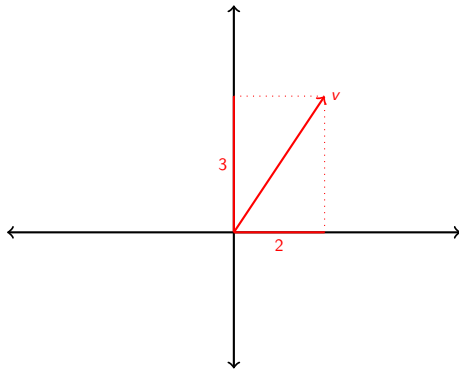
Notation is important. For this set of slides consider:

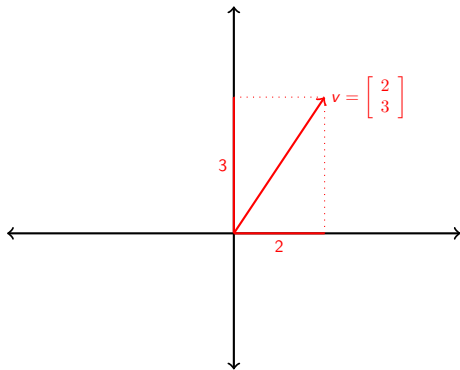
1. Lowercase for elements of a *vector*, v_i .
2. Uppercase for vectors/matrices, V .
3. Calligraphic uppercase for sets, e.g., set \mathcal{S} .

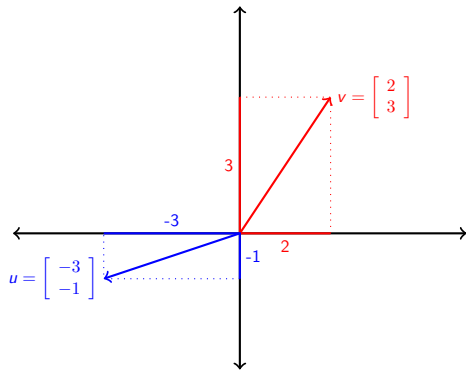


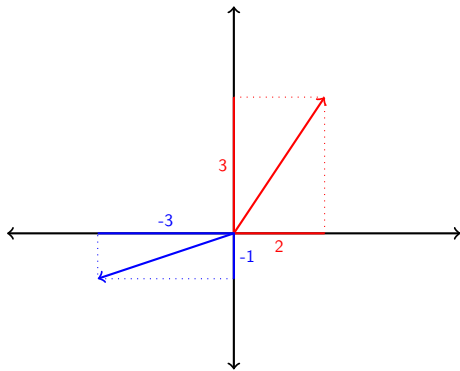




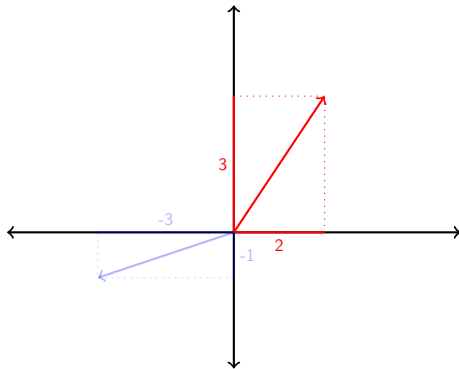


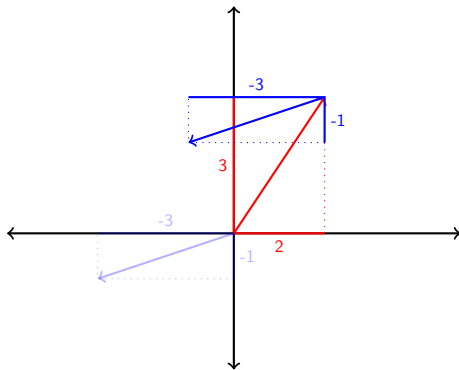


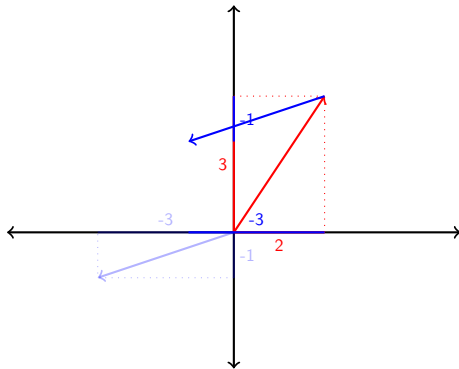


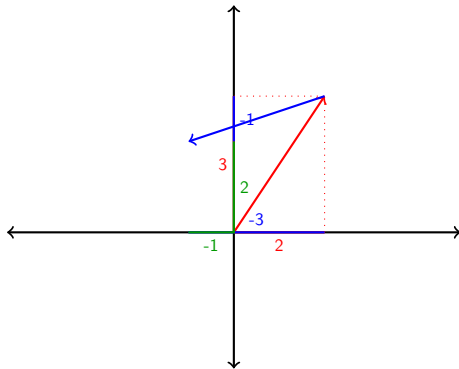


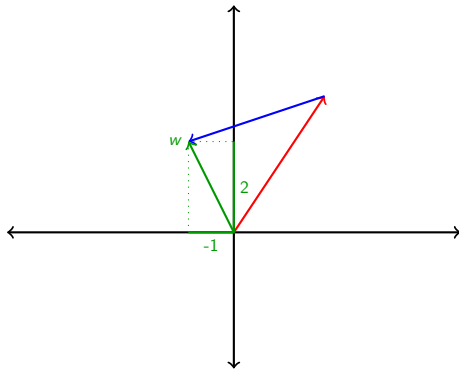
$$u + v?$$

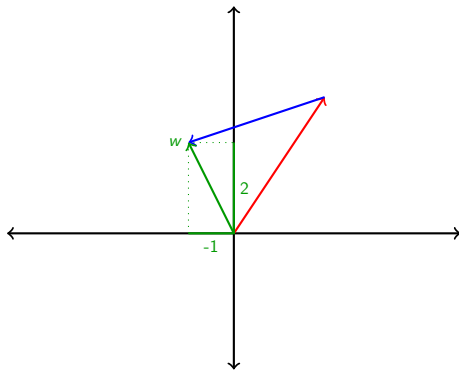




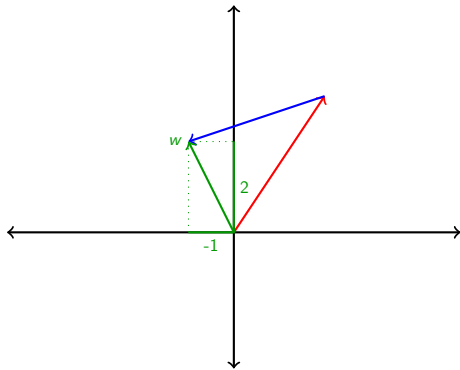






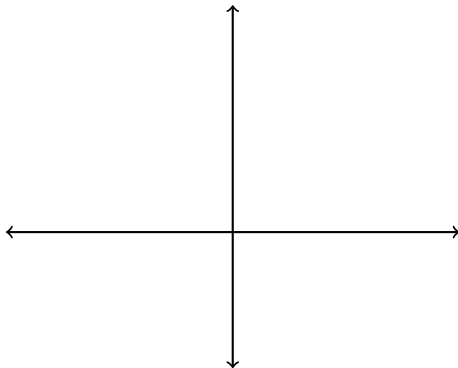


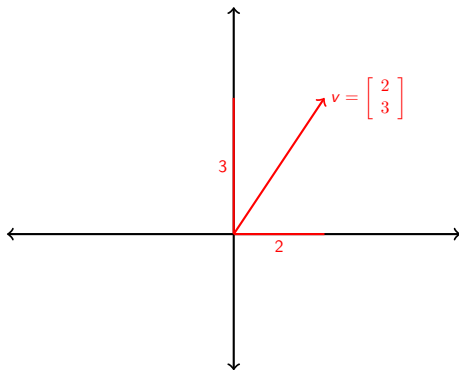
$$u + v = w$$

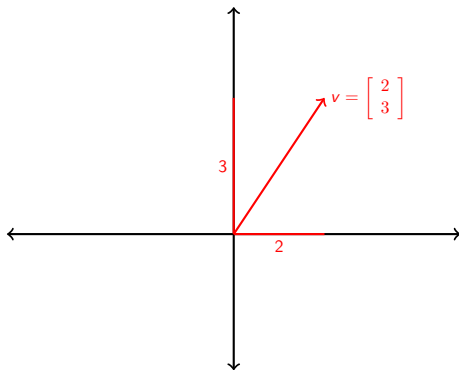


$$u + v = w$$

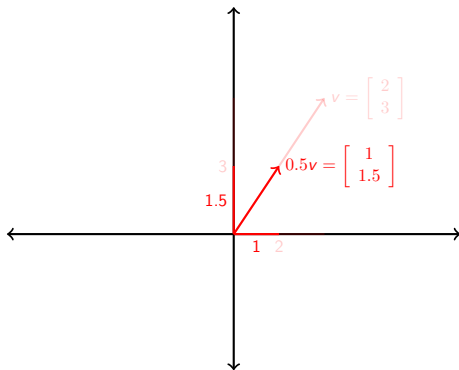
$$\begin{bmatrix} 2 \\ 3 \end{bmatrix} + \begin{bmatrix} -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

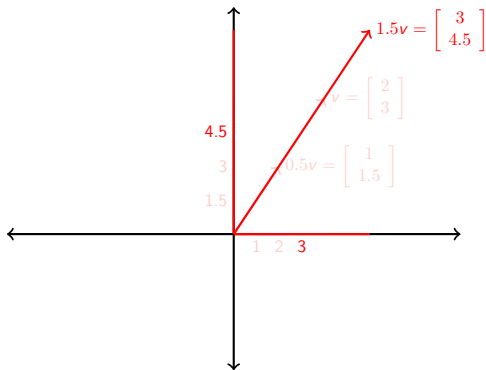


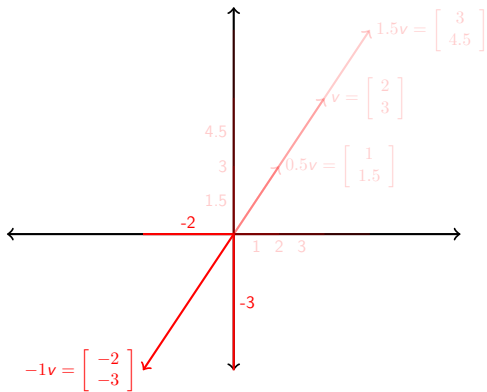


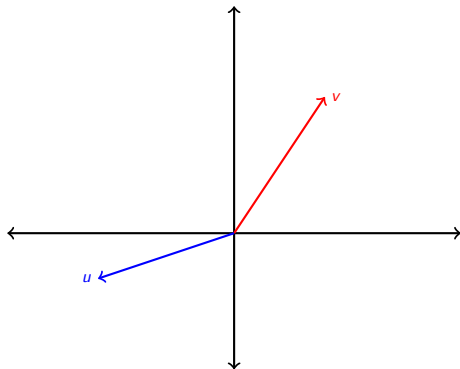


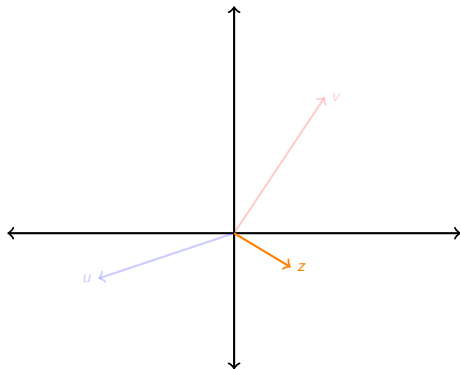
$0.5v?$

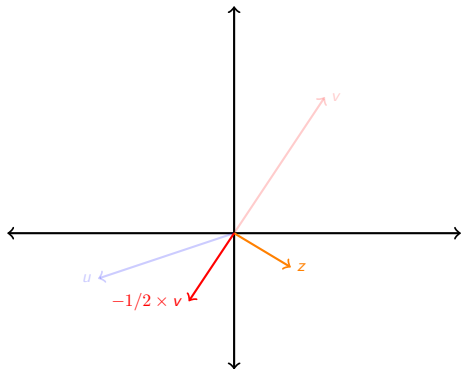


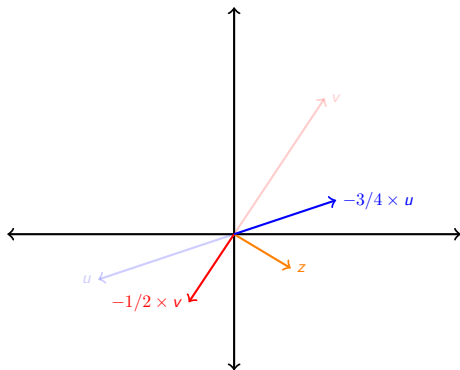


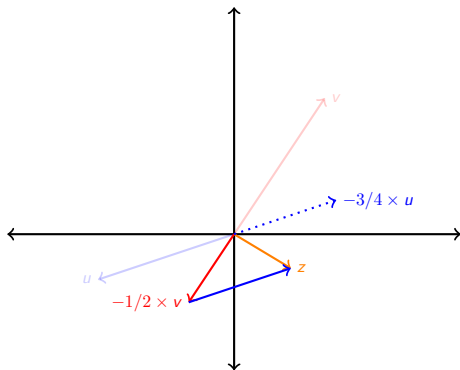


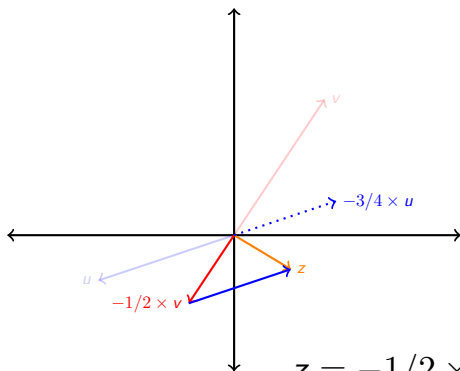




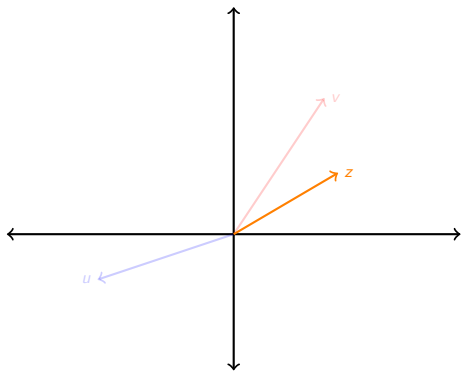


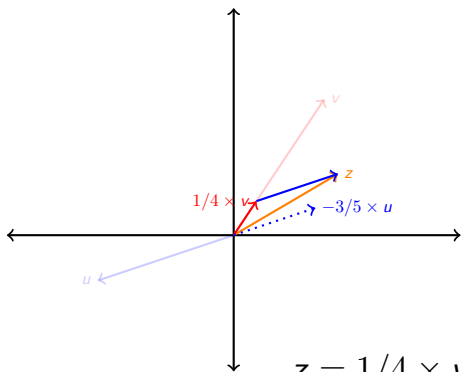




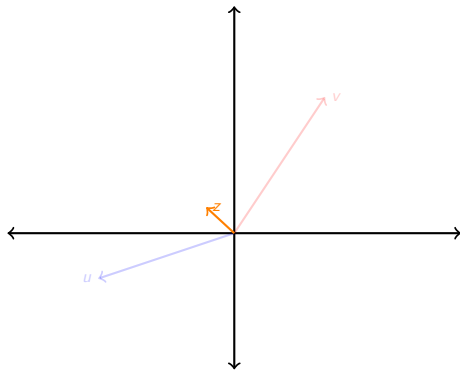


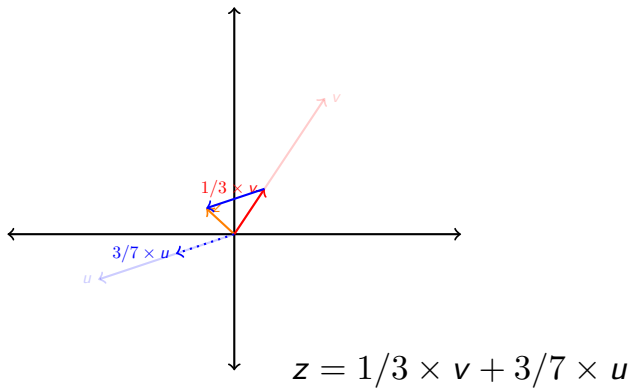
$$z = -1/2 \times v + -3/4 \times u$$





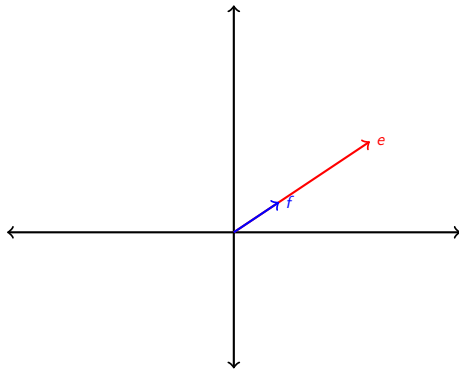
$$z = 1/4 \times v + -3/5 \times u$$



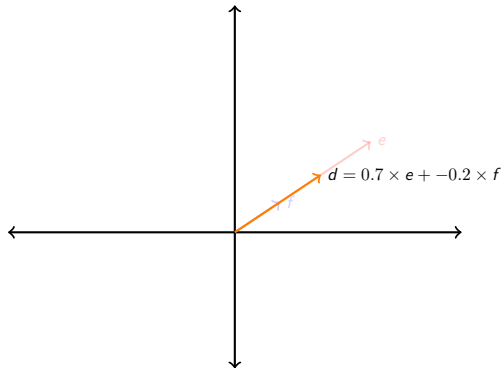


We can write any vector in the plane as the result of the product and sum of u and v (a.k.a. a *linear combination*). These vectors, are not special, except for 1 thing... they are linearly independent.

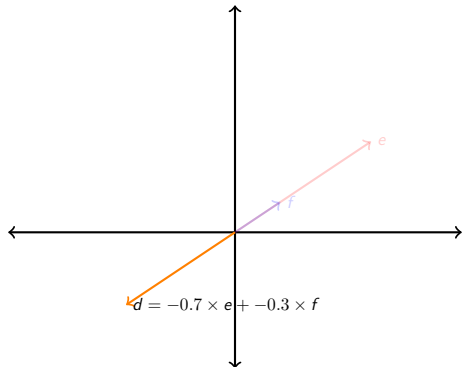
Consider now these two vectors e and f ...



Consider now these two vectors e and f ...

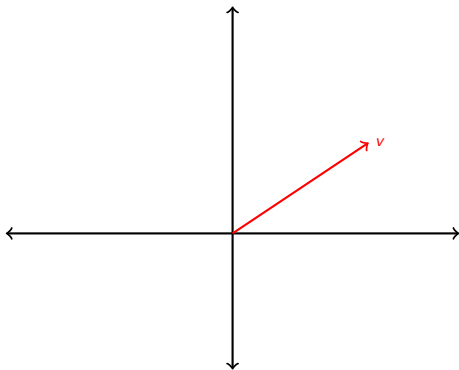


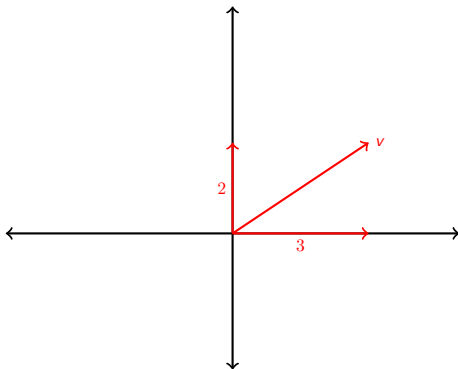
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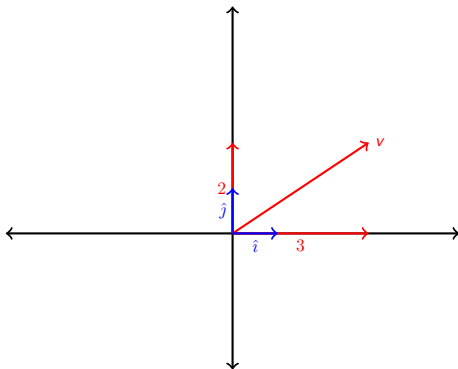


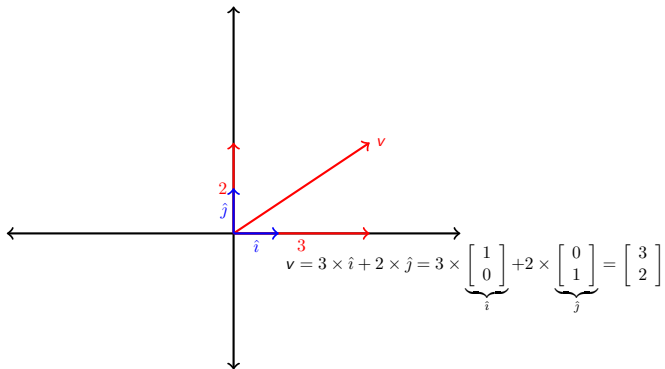
We can only “create” vectors along the same line, the line that goes in the direction of vectors e and f . These vectors are linearly dependent.

Actually, we only needed one of them to create all the others that we could draw!









Definition

A **vector** is an element V of \mathbb{R}^n , for $n \geq 2$. A scalar is an element of \mathbb{R} .

Vectors are to be written as columns, example:

$$V = \begin{pmatrix} v_1 \\ v_2 \\ \dots \\ v_{n-1} \\ v_n \end{pmatrix} \in \mathbb{R}^n$$

Let $X, Y \in \mathbb{R}^n$, and $\alpha \in \mathbb{R}$, then

1. The sum,

$$X + Y = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{pmatrix}$$

2. Scalar multiplication,

$$\alpha X = \begin{pmatrix} \alpha x_1 \\ \vdots \\ \alpha x_n \end{pmatrix}$$

$0 \in \mathbb{R}^n$ is,

$$\begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

that is a vector of dimension $n \times 1$ filled with zeroes.

Definition

A **vector space** \mathcal{S} , satisfies that, for any $A, B \in \mathcal{S}$, and $\alpha \in \mathbb{R}$,

- $(A + B) \in \mathcal{S}$
- $\alpha A \in \mathcal{S}$

It is trivial to show that \mathbb{R}^n is a vector space.

Definition

A nonempty set $\mathcal{S} \subseteq \mathbb{R}^n$ is a **vector subspace** of \mathbb{R}^n if, with the vector addition and the scalar multiplication it is a vector space by itself.

Conjecture

Let $\mathcal{V} \subseteq \mathbb{R}^n$, nonempty. \mathcal{V} is a vector subspace of \mathbb{R}^n if and only if,

- 1. $0 \in \mathcal{V}$,*
- 2. $a, b \in \mathcal{V}$, $\alpha \in \mathbb{R}$, then $a + \alpha b \in \mathcal{V}$*

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Quick quiz! 15 min to prove it!

Proof.

- $\Rightarrow \dots$ If \mathcal{V} is v.s. of \mathbb{R}^n , we know that the scalar multiplication and the sum is in the space. Because scalar mult. we know that $\alpha b \in \mathcal{V}$, so the sum must be in \mathcal{V} too.

Proof.

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- \Leftarrow ... If $a + \alpha b \in \mathcal{V}$, then it holds in particular for $\alpha = 1$, so the sum is *closed* in the space. Also, let $a = 0$, and you have the scalar multiplication. Then \mathcal{V} must be a v.s.



Definition

Let $\mathcal{V} \subseteq \mathbb{R}^n$ be a set of k vectors, then, $Z \in \mathbb{R}^n$ is a **linear combination** of the vectors $\{V_i\}_{i=1}^k$ in \mathcal{V} if there are scalars α_j $j = 1, \dots, k$ such that,

$$Z = \sum_{j=1}^k \alpha_j V_j$$

Definition

A **linear subspace** generated by the vectors in \mathcal{V} , represented $L(\mathcal{V})$, is the set of all the linear combinations of those vectors.

Conjecture

1. *Let $\mathcal{V}, \mathcal{W} \subseteq \mathbb{R}^n$, such that $\mathcal{V} \subseteq \mathcal{W}$, then $L(\mathcal{V}) \subseteq L(\mathcal{W})$*
2. *If $Y \in L(\mathcal{V})$, then $L(\{Y\} \cup \mathcal{V}) = L(\mathcal{V})$*
3. *Given a nonempty $\mathcal{V} \subseteq \mathbb{R}^n$, then $L(\mathcal{V})$ is a vector subspace of \mathbb{R}^n .*

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Quick quiz! Prove it \rightarrow 15 min.

Proof.

1. Trivial. If $X \in L(\mathcal{V}) \Rightarrow X = \sum_{v_i \in \mathcal{V}} \alpha_i v_i$, and because $\mathcal{V} \subseteq \mathcal{W}$ those vectors are also part of \mathcal{W} , so $X \in L(\mathcal{W})$, so $L(\mathcal{V}) \subseteq L(\mathcal{W})$.

Proof.

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2. ■ As $\mathcal{V} \subseteq \mathcal{V} \cup \{Y\}$, we have that $L(\mathcal{V}) \subseteq L(\mathcal{V} \cup \{Y\})$.

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 - Let $X \in L(\mathcal{V} \cup \{Y\})$, then there are scalars α_i such that

$$X = \sum_{v_i \in \mathcal{V}} \alpha_i v_i + \beta Y$$

Proof.

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- As $Y \in L(\mathcal{V})$ there are scalars γ_i such that $Y = \sum_{v_i \in \mathcal{V}} \gamma_i v_i$

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- $X = \sum_{v_i \in \mathcal{V}} \alpha_i v_i + \beta \left(\sum_{v_i \in \mathcal{V}} \gamma_i v_i \right) = \sum_{v_i \in \mathcal{V}} (\alpha_i + \beta \gamma_i) v_i$. But $\alpha + \beta \gamma$ is a scalar, so

Proof.

1. Trivial. If $X \in L(\mathcal{V}) \Rightarrow X = \sum_{v_i \in \mathcal{V}} \alpha_i v_i$, and because $\mathcal{V} \subseteq \mathcal{W}$ those vectors are also part of \mathcal{W} , so $X \in L(\mathcal{W})$, so $L(\mathcal{V}) \subseteq L(\mathcal{W})$.
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- $X \in L(\mathcal{V})$, proof is complete.

Proof.

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 - Let $X \in L(\mathcal{V} \cup \{Y\})$, then there are scalars α_i such that

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 - $X \in L(\mathcal{V})$, proof is complete.
3. 0 belongs to any $L()$, as it is the case with scalars = 0. Now, let $X, Y \in L(\mathcal{V})$ and $\gamma \in \mathbb{R}$; $X + \gamma Y = \sum_{v_i \in \mathcal{V}} (\alpha_i + \gamma \beta_i) v_i$ if we write each vector as a linear comb. For the same argument used before, we complete the proof.



Definition

A set of k vectors $\mathcal{V} \subseteq \mathbb{R}^n$ is **linearly independent** if, $\forall \alpha_j \in \mathbb{R}$

$$\sum_{j=1}^k \alpha_j V_j = 0 \quad \Leftrightarrow \quad \alpha_j = 0$$

Definition

Conversely, if there are $\{\alpha_i\}_{i=1}^k$, with at least one $\alpha_k \neq 0$, then they are **linearly dependent**.

Definition

The set of vectors $\mathcal{X} \subseteq \mathcal{V}$ **generates** the vector subspace $\mathcal{V} \subseteq \mathbb{R}^n$ if any $V \in \mathcal{V}$ can be written as a linear combination of the vectors in \mathcal{X} .

Moreover, if the vectors in \mathcal{X} are linearly independent, then \mathcal{X} is called a **basis** of \mathcal{V} .

Conjecture

Let $\mathcal{X} = \{X_1, X_2, \dots, X_k\}$ a basis of the vector subspace $\mathcal{V} \subseteq \mathbb{R}^n$. Then, for any $V \in \mathcal{V}$, there are **unique** scalars $\{\alpha_i\}_{i=1}^k$ such that

$$V = \alpha_1 X_1 + \alpha_2 X_2 + \dots + \alpha_k X_k$$

Conjecture

Any set of n linearly independent vectors $\mathcal{X} \subseteq \mathbb{R}^n$, generates \mathbb{R}^n

Definition

The **dimension** of a vector space \mathcal{V} is the maximum number of *l.i.* vectors that generates it. This number coincides with the number of vectors in any basis of the space. It is denoted $\dim(\mathcal{V})$.

Definition

Given $X, Y \in \mathbb{R}^n$, the **inner product** corresponds to:

$$X \cdot Y = \sum_{j=1}^n x_j y_j \in \mathbb{R}$$

Definition

The **Euclidean norm** of a vector $X \in \mathbb{R}^n$ is:

$$\|X\| = \sqrt{X \cdot X} = \sqrt{\sum_{j=1}^n x_j^2} \in \mathbb{R}$$

Definition

For $X, Y \in \mathbb{R}^n$, the **Euclidean distance** between them is defined as:

$$d(X, Y) = ||X - Y||$$

Definition

For $X, Y \in \mathbb{R}^n$, both different from zero, the **angle** between them, denoted as $\angle(X, Y)$, is defined as the value that satisfies,

$$\cos(\angle(X, Y)) = \frac{X \cdot Y}{\|X\| \cdot \|Y\|} \in [-1, 1]$$

Definition

Two vectors X, Y are **orthogonal**, if $\angle(X, Y) = 90^\circ$, or equivalently, $X \cdot Y = 0$. It is denoted as $X \perp Y$.

Definition

Let $X \in \mathbb{R}^n$. If $\|X\| = 1$, X is a **unit vector**.

Definition

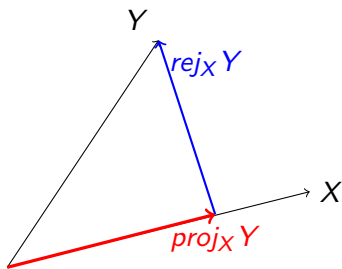
Consider two vectors $X, Y \in \mathbb{R}^n$, both different from zero. The **projection** of Y over X is defined as:

$$proj_X Y = Y \cdot \frac{X}{\|X\|}$$

The **rejection**, is defined as:

$$rej_X Y = Y - proj_X Y$$

The rejection is orthogonal to X .

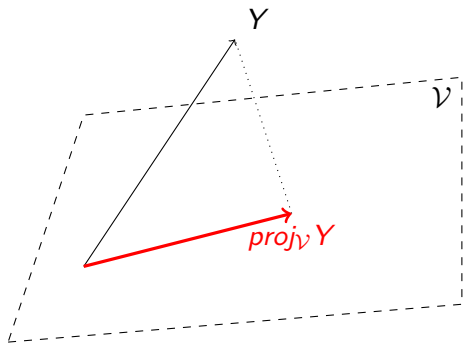


In econometrics, the endogenous variable would be Y . We try to explain it with the exogenous variable X , so we “project” Y over X . Of course, what is not explained, the error, is $rej_X Y$.

Definition

The **projection** of a vector Y over a subspace \mathcal{V} defined by a basis $\{X_1, X_2, \dots, X_k\}$ is the vector $proj_{\mathcal{V}} Y$, and it must satisfy that

$$[proj_{\mathcal{V}} Y - Y] \perp X_i \quad \forall i = 1, \dots, k$$



Here we could be projecting the exogenous variable over two explanatory variables...